

## Dynamics of three-agent games

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2008 J. Phys. A: Math. Theor. 41 055002

(<http://iopscience.iop.org/1751-8121/41/5/055002>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.152

The article was downloaded on 03/06/2010 at 07:22

Please note that [terms and conditions apply](#).

# Dynamics of three-agent games

Muhittin Mungan<sup>1,2</sup> and Tonguç Rador<sup>1</sup>

<sup>1</sup> Department of Physics, Boğaziçi University, Bebek 34342, Istanbul, Turkey

<sup>2</sup> The Feza Gürsey Institute, PO Box 6, Çengelköy, 34680 Istanbul, Turkey

E-mail: [mmungan@boun.edu.tr](mailto:mmungan@boun.edu.tr) and [tonguc.rador@boun.edu.tr](mailto:tonguc.rador@boun.edu.tr)

Received 7 November 2007

Published 23 January 2008

Online at [stacks.iop.org/JPhysA/41/055002](http://stacks.iop.org/JPhysA/41/055002)

## Abstract

We study the dynamics and resulting score distribution of three-agent games where after each competition a single agent wins and scores a point. A single competition is described by a triplet of numbers  $p$ ,  $t$  and  $q$  denoting the probabilities that the team with the highest, middle or lowest accumulated score wins. The three-agent game can be regarded as a social model where a player can be favored or disfavored for advancement, based on his/her accumulated score. We study the full family of solutions in the regime, where the number of agents and competitions is large, which can be regarded as a hydrodynamic limit. Depending on the parameter values  $(p, q, t)$ , we find six qualitatively different asymptotic score distributions and we provide a qualitative explanation of these results. We also compare our analytical results against numerical simulations of the microscopic model and find these to be in excellent agreement. It is possible to decide the outcome of a three-agent game through a mini-tournament of two-agent competitions among the participating players and it turns out that the resulting possible score distributions are a subset of those obtained for the general three-agent games. We discuss how one can add a steady and democratic decline rate to the model and present a simple geometric construction that allows one to obtain the score evolution equations for  $n$ -agent games.

PACS numbers: 02.50.-r, 87.23.Ge, 05.40.-a

## 1. Introduction

The use of physical-mathematical modeling to describe phenomena involving biological, social, political, economical and other systems has grown into an increasingly active area of cross-disciplinary research. Often these systems share common aspects: a collection of agents and a set of rules to describe their interaction [1–3]. A reasonable approach to study such interactions is to involve a finite number of agents at a time, cast their interactions in

terms of a competition for a utility and determine the rules to decide the winner to gain one unit of this attribute in a competition. Thus the competition might be for wealth [4–7], trophies in sports [8, 9], opinion dynamics [10–12], or the propagation of ideas and rumors [13–15]. One can also address the emergence of social hierarchies from such models [16–20]. On the other hand, there are also settings in which the interaction between agents is not purely competitive and where the weaker players might have a higher chance of winning. Such models may admit solutions that could be useful for making analogies with welfare systems, for example.

Ben-Naim et al have recently extended a two-agent game model [16–20] to three-agent interactions [21], where a single competition involves three individuals at a time. The winner in a competition is determined by three numbers  $(p, t, q)$  describing the winning probability of the agent with largest, middle and lowest points, respectively. This model was studied in detail for the three special cases  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$  [21]. These parameter choices correspond to the cases where in a given competition the agent with the largest, middle or lowest score wins a given competition with certainty one, respectively. The aim of the present paper is to study the model for the entire phase space of parameters, bounded only by the conservation of probability,  $p + t + q = 1$ . By applying the theory of weak solutions to hyperbolic conservation laws, we obtain the full set of solutions whose richness goes well beyond the three special cases considered in [21] and also has a much richer solution structure than the two-agent game. Three-agent games can also be motivated by the observation that in social interactions generally more than two agents are involved, such as, in a job application or companies competing for a contract.

The paper is organized as follows. In section 2 we present the model and its mathematical interpretation, yielding the equations governing the score distribution of the agents. In section 2 we also present a simple geometric method to generalize the model to  $n$ -agent competitions. Section 3 constitutes the main part of the paper. We introduce the method of characteristics and derive the form of the weak solutions for the equation governing the evolution of the cumulative score distribution obtained in section 2. Using these results, we then obtain the full family of score distribution functions. We next interpret the solutions found in terms of score-based social structures in a society of agents. In section 4 we consider two variations of the three-agent model: (i) the resolution of a subset of three-agent games in terms of a mini-tournament involving only two-agent interactions, and (ii) the introduction of a decline rate for the agents to describe the loss of fitness due to inactivity, as advocated in [20]. The last section is reserved for a short discussion and comments on further interesting extensions of such models.

## 2. The model

In the three-agent game a competition is described as follows [21]: we first pick three out of a collection of  $N$  agents. We advance only one of the agents, based on their accumulated points prior to the competition. Let the three agents have scores  $x$ ,  $y$  and  $z$ , respectively. Then, the agent with the largest score will increase its score by one unit with probability  $p$ , the one with the smallest score with probability  $q$  and the one in the middle with probability  $t$ . The situation with equal scores is evaluated on the basis of equal likelihood. The initial condition is such that at the beginning all agents start with a zero score.

Assuming an ordering of points is made so that one has  $x \geq y \geq z$ , we can cast the rules of the competition as

$$(x > y > z) \implies (p, t, q) \tag{1a}$$

$$(x = y > z) \implies \left( \frac{p+t}{2}, \frac{p+t}{2}, q \right) \tag{1b}$$

$$(x > y = z) \implies \left( p, \frac{t+q}{2}, \frac{t+q}{2} \right) \tag{1c}$$

$$(x = y = z) \implies \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right). \tag{1d}$$

Here the lists on the right represent the winning probabilities of the teams with points listed on the left. There are no ties so that after every game one agent surely advances its score, and therefore

$$p + t + q = 1. \tag{2}$$

From this microscopic model we can read out the change of the number of agents in a particular score range. Let us denote by  $f_x$  the fraction of agents having score  $x$ . After a competition some agents might leave this region and some might enter it by winning a competition in each case. This implies the following local conservation law:

$$\frac{df_x}{d\tau} = \sum_{y,z} (f_{x-1} W_{x-1,y,z} - f_x W_{x,y,z}) f_y f_z. \tag{3}$$

Here  $W_{x,y,z}$  denotes the probability that agents with  $x$  points will win in a competition against two other agents with points  $y$  and  $z$ . Note that the microscopic rules in equation (1) completely define  $W_{x,y,z}$ . The right-hand side is cubic in  $f$ , since we are picking three agents out of the collection, and the probability of picking an agent with a given score  $x$  is  $f_x$ .

Since

$$\sum_x f_x = 1, \tag{4}$$

it is immediate that equation (3) also implies, as it should, the global conservation of the total number of agents, as can be checked by performing a sum over  $x$ , with  $x \geq 0$ , and setting  $f_{-1} = 0$ , since agents cannot have negative scores in our model.

The time variable  $\tau$  in equation (3) has an arbitrary scaling which can be compensated by an overall factor in the definition of  $W$ . The natural scale is such that the average points of agents is given by

$$\bar{x}(\tau) \equiv \sum_{x=0}^{\infty} x f_x = \frac{\tau}{3}, \tag{5}$$

meaning that (on average) each agent participates in a single game during each round  $\tau$  of games and where only one of the participating agents in a game wins.

The presence of sums over the discrete indices on the right-hand side of equation (3), results in a coupled set of differential equations. These can be further simplified by defining

$$F_x \equiv \sum_{y=0}^{x-1} f_y \tag{6}$$

so that

$$f_x = F_{x+1} - F_x. \tag{7}$$

Summing (3) over  $x$ , the sum telescopes out and we obtain

$$\frac{dF_x}{d\tau} = -f_{x-1} \sum_{y,z} W_{x-1,y,z} f_y f_z, \tag{8}$$

since the surface term at  $x = -1$  is set to zero.

Note that

$$f_{x-1} \sum_{y,x} W_{x-1,y,z} f_y f_z \tag{9}$$

yields the probability that an agent with score  $x - 1$  will win a competition against any two agents.

Equation (8) has a simple interpretation: since agents cannot decrease their scores, agents within a score range  $[0, x - 1]$  can only move out of this range, while no agent can enter it. Thus the number of agents in this range,  $F_x$  can only decrease or at best remain constant from competition to competition, thereby accounting for the single loss term on the right-hand side of equation (8). Furthermore, at any round of games only those agents with score  $x - 1$  can leave this range by winning a game and the probability for this is given by equation (9), the RHS of equation (8).

The discussion leading from the conservation law equation (3) to the cumulative score evolution equation, (8) is applicable to any kind of rule for determining the outcome of a three-agent competition as long as it is entirely score based. For the particular rules, equation (1), the form of equation (8) can be worked out easily, yielding

$$\begin{aligned} \frac{dF_x}{d\tau} = & -f_{x-1} [pF_{x-1}^2 + q(1 - F_x)^2 + 2tF_{x-1}(1 - F_x)] \\ & - 2\frac{(p+t)}{2} f_{x-1}^2 F_{x-1} - 2\frac{(t+q)}{2} f_{x-1}^2 (1 - F_x) - \frac{1}{3} f_{x-1}^3. \end{aligned} \tag{10}$$

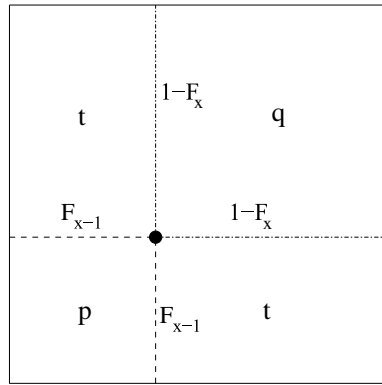
The terms on the RHS of the evolution equation describe the different line-ups under which an agent with score  $x - 1$  can score a point and move out of the score range  $[0, x - 1]$ . The terms on the first line of equation (10) represent the bulk of the interactions and involve three players with different scores. They constitute the rule of equation (1a) in which the agent with score  $x - 1$  is in turn the competitor with the highest, middle or lowest score in the line-up. We will refer to the remaining terms on the RHS of equation (10) as interface interactions: the second and third lines represent the cases where two players have identical scores  $x - 1$ , rules equations (1b) and (1c) respectively, and the last term represents the case when all the teams have equal score, equation (1d).

In a microscopic simulation of the system the interface terms dominate only at early times, since at the beginning all teams have zero points. In fact, the most important contribution at the very beginning is described by the  $f_{x-1}^3$  term in equation (10). The effect of this term is to create random imbalances in the point distribution which in time will make the  $f_{x-1}^2$  terms more relevant. These imbalances are further emphasized by the dynamics of the bulk terms as prescribed by the set of winning probabilities  $(p, t, q)$ . Therefore as time goes by, in a thermodynamic limit where the number of teams ranges to infinity, the majority of the contributions to the dynamics will be governed by the bulk terms, corresponding to three-agent competitions among players with unequal scores. On the other hand as time goes by, every team will accumulate a certain number of points which, in general, will be larger than a single point. These considerations allow one to go to a continuum limit where the function  $F_{x-1}$  can be expanded in terms of the derivatives at  $x$ . A first order approximation, where one considers only the bulk terms, results in the following,

$$\frac{\partial F}{\partial \tau} = -\frac{\partial F}{\partial x} G'(F), \tag{11}$$

with

$$G'(F) \equiv pF^2 + 2tF(1 - F) + q(1 - F)^2. \tag{12}$$



**Figure 1.** The geometric representation of the sum in equation (9). The filled circle represents the team with point  $x - 1$ . The dashed and dot-dashed lines represent the number of teams with points strictly less or greater than  $x - 1$ . The sum over the areas will bring the bulk interaction whereas the sum over the interfaces represent the cases where two teams have equal points. The dot gives the term when all the teams have equal points. This construct is easily generalized to  $n$ -agent games as discussed in the text.

As shown in [20], the contributions of the higher order derivative terms, as well as the interface terms, become negligible in the asymptotic time limit (see also section 3, for numerical evidence) and we will refer to this as the hydrodynamical limit.

The fact that in the hydrodynamical limit the interface terms become negligible is also corroborated by the fact that they do not contribute to the asymptotic time dependence of the average score. We have

$$\frac{d\bar{x}(\tau)}{d\tau} = \frac{d}{d\tau} \int_0^\infty dx x f = - \int_0^\infty dx \frac{\partial F}{\partial \tau}, \tag{13}$$

and using equation (11) we find

$$\frac{d\bar{x}(\tau)}{d\tau} = \int_0^\infty dx \frac{\partial F}{\partial x} G'(F). \tag{14}$$

Noting that  $F(x = \infty) = 1$  and that in the large time limit  $F(x = 0) = 0$  (since a team with zero points will happen with vanishing probability in the limit of large times) we obtain

$$\frac{d\bar{x}(\tau)}{d\tau} = \int_0^1 dF G'(F) = \frac{p + t + q}{3} = \frac{1}{3}, \tag{15}$$

which implies that asymptotically  $\bar{x}(\tau) = \tau/3$ .

### 2.1. Generalization to $n$ -agent games

The terms on the right-hand side of equation (10) can also be obtained by invoking a geometric construction that readily lends itself to a generalization to  $n$ -agent games. Imagine all the agents arranged on a line ordered by their discrete scores. Let two perpendicular axes formed in this way represent the possible opponents that a player with score  $x - 1$  can have in the competition. The situation is represented graphically in figure 1, where the filled circle represents a player with score  $x - 1$ . The dashed and dot-dashed lines represent the boundaries for the cases where an opponent's score is strictly less or greater than  $x - 1$ , respectively. The sum over the areas constitutes the bulk of the match-ups in which all three players have different scores, e.g.

there are  $F_{x-1}^2$  agents with scores less than  $x - 1$  and thus the agent with score  $x - 1$  will win such games with probability  $p$ . The sum over the interfaces (dashed line segments) represent the cases where the agent with score  $x - 1$  and one of its opponents have equal scores. Thus for example there are  $f_{x-1} F_{x-1}$  opponents such that one of them has score  $x - 1$  and the other has a lower score, so that one of the agents with score  $x - 1$  wins the game with probability  $(p + t)/2$ . The dot represents the case when all three agents have identical score  $x - 1$ .

This geometric construction can now be generalized to  $n$ -agent games. For simplicity, we will consider the form of the equation only in the hydrodynamical limit, where the bulk terms dominate the score evolution. The interface terms can be generated in an analogous manner. The construction is simply picking a point in an  $(n - 1)$ -dimensional cube and splitting the cube's volume by  $(n - 2)$ -dimensional planes passing through it, which are orthogonal to each other and also to the sides of the cube. This construction leads to a division of the cube into  $2^{n-1}$  volumes, and as is readily shown, the number of equivalent  $(n - 1)$ -volumes is given by the binomial coefficients. The dynamics of a  $n$ -agent game, in the hydrodynamical limit, is therefore again described by an equation of the form

$$\frac{\partial F}{\partial \tau} = -\frac{\partial F}{\partial x} G'(F), \tag{16}$$

with

$$G'(F) = \sum_{k=0}^{n-1} \binom{n-1}{k} p_k (1-F)^k F^{n-1-k}, \tag{17}$$

and where  $p_k$  is the probability for the player with the  $k$ th highest score to win a competition, obeying

$$\sum_{k=0}^{n-1} p_k = 1. \tag{18}$$

The mean score thus evolves as

$$\bar{x}(\tau) = \frac{\tau}{n}, \tag{19}$$

as is readily shown.

### 3. Analysis of the model

We note that equation (11) is in the form of a hyperbolic conservation law and apply the theory of weak solutions, also known as the *method of characteristics*. In the next subsection we therefore turn first to a brief description of the method of characteristics and then proceed to obtain the solution of equations (11) and (12) for the full parameter range  $p + q + t = 1$ . For a more detailed account of the theory of hyperbolic PDEs and conservation laws, we refer the reader to [22, 23].

#### 3.1. The method of characteristics

We are looking for a solution of

$$\frac{\partial F}{\partial \tau} = -\frac{\partial F}{\partial x} G'(F), \tag{20}$$

subject to the initial condition

$$F(x, 0) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases} \tag{21}$$

Given (20), we define its characteristics as the curves  $x(\tau)$  in the  $x - \tau$  plane on which  $F(x, \tau)$  remains constant. It can be shown that these curves are given by the *characteristic equation*

$$x(t) = x_0 + \tau G'(F(x_0, 0)) \tag{22}$$

and thus  $G'(F(x_0, 0))$  is the speed of the characteristic emerging from the point  $x_0$ .

An implicit solution can therefore be found as

$$F(x, \tau) = F(x_0, 0), \tag{23}$$

where for a given  $(x, \tau)$ ,  $x_0$  is determined from the characteristics, equation (22).

Depending on the initial conditions and the form of  $G'(F)$ , the characteristics can intersect thereby giving rise to multiple-valued points that are resolved by introducing discontinuities (shocks). Even with smooth initial data, discontinuities can develop in a finite time [22, 23]. Since solutions with discontinuities do not form a strict solution of the partial differential equation, one denotes these as weak solutions which instead of the local PDE are required to obey a weaker form of the conservation law,

$$\int_0^\infty d\tau \int_{-\infty}^\infty dx \chi(x, \tau) \left[ \frac{\partial F}{\partial \tau} + G'(F) \frac{\partial F}{\partial x} \right] = 0, \tag{24}$$

for any continuously differentiable function  $\chi(x, \tau)$  with compact support [22, 23]. Note that one can associate with equation (20) an infinite family of equations,

$$\frac{\partial U(F)}{\partial \tau} = - \frac{\partial R(F)}{\partial x}, \tag{25}$$

by setting

$$R'(F) = G'(F)U'(F). \tag{26}$$

If  $F(x, \tau)$  has discontinuities, and hence is not differentiable everywhere, each choice of  $U(F)$  will lead to a different integral form of the conservation law, equation (24), and hence to a different weak solution [22]. Thus the desired weak solution has to be determined from the appropriate conservation law valid also in the presence of discontinuities. In general, this requires inspecting the microscopic evolution from which the continuum description arose. For the present case, the appropriate form of the conservation law is readily determined by

$$\frac{d}{d\tau} \int_0^\infty F(x, \tau) d\tau = - G(F)|_0^1, \tag{27}$$

which from equations (13) and (14) is seen to be equivalent to equation (15), simply stating that the average score increases at a rate 1/3, independent of the score distribution, since on average only one out of three players scores a point in a given game.

Note that our initial conditions, equation (21) contains a discontinuity. The weak solution described above prescribes how these are handled, as follows. Given a discontinuous segment of  $F$  at  $x_0$ , such that  $F_l$  and  $F_r$  are the values of  $F$  immediately to the left and right of the discontinuity, the speed of the characteristics are given as  $G'(F_l)$  and  $G'(F_r)$ , respectively. There are two cases that one needs to distinguish: (i)  $G'(F_l) > G'(F_r)$ , and (ii)  $G'(F_l) < G'(F_r)$ . In the former case we have a moving shock discontinuity, while in the latter case, we have a *rarefaction wave*.

- (i)  $G'(F_l) > G'(F_r)$ . Applying the integral form of the conservation law around the discontinuity, it can be shown that the shock moves with a speed [22]

$$v = \frac{G(F_l) - G(F_r)}{F_l - F_r} = \frac{\Delta G}{\Delta F}. \tag{28}$$

This is known as the Rankine–Hugoniot jump condition [23].



(ii)  $G'(F_l) < G'(F_r)$ . In this case the characteristics immediately to the left and right of the discontinuity diverge from each other. The weak solution in this case is given by a similarity solution [22, 23]

$$F(x, \tau) = \Phi(x/\tau), \quad (29)$$

for  $(x, \tau)$  such that

$$G'(F_l) < \frac{x - x_0}{\tau - \tau} < G'(F_r), \quad (30)$$

and where  $\Phi(z)$  is given implicitly by

$$G'(\Phi) = z. \quad (31)$$

*3.1.1. The scaling Ansatz.* The existence of a smooth scaling solution of the form equation (29) is a direct consequence of the evolution by characteristics, equations (22) and (23). Equation (31) can therefore also be obtained from substituting the Ansatz, equation (29), into the PDE, equation (20) to obtain

$$\frac{d\Phi}{dz}[-z + G'(\Phi)] = 0, \quad (32)$$

from which the solutions  $\Phi = \text{constant}$  and equation (31) follow. The approach taken in [20, 21] has been to combine both kinds of solutions in order to obtain the full solution. In fact, the possibility to do so already suggests that the derivative of  $F$  need not be continuous and even  $F$  itself need not be continuous anymore. It is worth re-emphasizing that the contribution of the interface and hence higher order derivative terms will tend to smooth out these discontinuities at early times, but that asymptotically their overall effects are negligible. In fact, it can be shown [20, 21] that for large times  $\partial^n F / \partial x^n$  decay as  $1/\tau^n$ . We refer the reader to [20, 21] where these effects have been studied in detail.

*3.1.2. Application to the two-agent game.* In order to illustrate the method of characteristics it is instructive to consider first the two-agent game. For the two-agent game we have [19, 20]

$$G'(F) = q + (p - q)F, \quad (33)$$

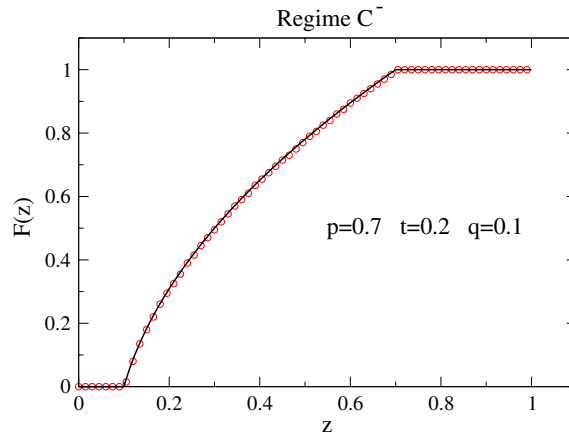
with  $p + q = 1$  and initial data given by equation (21). The speed of the characteristics to the left and right of the initial discontinuity at  $x = 0$  are  $G'(0) = q$  and  $G'(1) = p$ .

For  $p > q$  the characteristics spread out and we have  $F(x, \tau) = 1$  for  $x > p\tau$  and  $F(x, \tau) = 0$  for  $x < q\tau$ , while in the region  $q\tau \leq x \leq p\tau$  we have a rarefaction wave, equation (29) whose profile is found from equation (31) as

$$\Phi(z) = \frac{z - q}{p - q}. \quad (34)$$

The interpretation of the solution in terms of the underlying microscopical competition rules is that the strongest teams are becoming stronger, increasing their scores at a rate  $p$ , while the weakest teams can only increase their scores at a lower rate  $q$ . The score rate of the majority of teams lies in between these extreme cases and turns out to be uniformly distributed as a consequence that  $G'(F)$  is linear in  $F$ . The self-similar form of the score distribution in this regime is equivalent to a self-consistent mean-field solution to the microscopic evolution dynamics.

For  $p < q$ , the characteristics intersect. This means that once a team starts winning a series of games its increased score makes it less likely to win against most of the weaker teams, causing its score rate to decline. In the limit of very large scores, the average score



**Figure 2.** A representative graph for the solution in regime  $C^-$ . The circles represent the data from the simulation of the microscopic model. The solid line is the analytical solution.

rate of a team is  $1/2$  (since only half of the teams participating in a round of matches win) and fluctuations around this average become increasingly less (of the order of  $\tau^{-1/2}$ ), leading to the shock solution in the infinite time, *i.e. hydrodynamical*, limit.

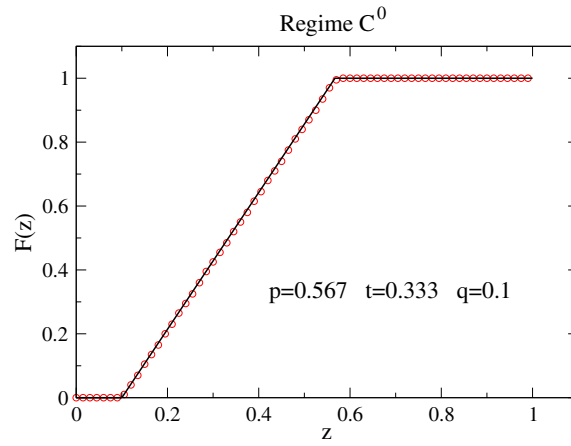
The shock speed is given by the jump condition, equation (28), with  $F_l = 0$  and  $F_r = 1$ , while  $G(F) = qF + (p - q)F^2/2$  so that we find indeed  $v = 1/2$ .

### 3.2. The three-player game

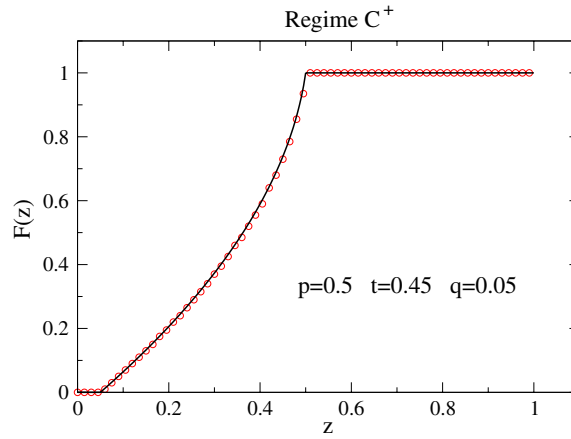
Depending on  $p$ ,  $t$  and  $q$ , the function  $G(F)$  is not necessarily convex in the interval  $F \in [0, 1]$  meaning that  $G'(F)$  is not a monotonically increasing or decreasing function which had been the case in the two-player game. As we have seen in the previous subsection, whenever a rarefaction wave  $G'(\Phi) = z$  is part of the solution one can expect the solution  $F$  to be concave up or down in  $x$ . When  $G'(F)$  is non-monotonous this implies that  $F$  will develop discontinuities, *i.e. shocks*. For general,  $p$ ,  $t$  and  $q$ , the solution of  $F$  turns out to contain a continuous segment given by the rarefaction profile  $G'(\Phi) = z$  as well as a single discontinuity. In summary, we find the following six regimes:

- $C^-$  :  $p > t \geq q$  and  $\tau < 1/3$ ,
- $C^0$  :  $t = 1/3 > q$ ,
- $C^+$  :  $t > 1/3$  and  $p \geq t$ ,
- $C_S^+$  :  $p < t$  and  $q \leq 1/3$ ,
- $S$  :  $q > 1/3 > p$ ,
- $C_S^-$  :  $q > t$  and  $p > 1/3$ .

In this suggestive notation, which will be justified below, regimes  $C^+$  and  $C^-$  represent a solution  $F$  which is concave up/down in  $x$  respectively. The regime  $C^0$  is the interface between the concave up and down solutions and has a vanishing second derivative. The label  $S$  means that there is a shock front in the solution. In particular, in the  $S$  regime the solution has the form of a single shock. That is,  $F$  is a step function. The corresponding profiles are shown in figures 2–7.



**Figure 3.** A representative graph for the solution in regime  $C^0$ . The circles represent the data from the simulation of the microscopic model. The solid line is the analytical solution.



**Figure 4.** A representative graph for the solution in regime  $C^+$ . The circles represent the data from the simulation of the microscopic model. The solid line is the analytical solution.

The solution in all six regions can thus be written in the general form

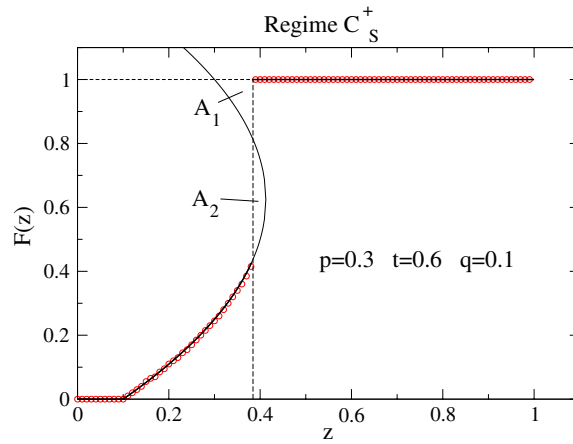
$$F(x, \tau) = \begin{cases} 0 & z < z_l, \\ \Phi(z) & z_l \leq z \leq z_r, \\ 1 & z > z_r, \end{cases} \quad (35)$$

where, again,  $z = x/\tau$ . We also define the roots  $\Phi_{\pm}(z)$  of the equation

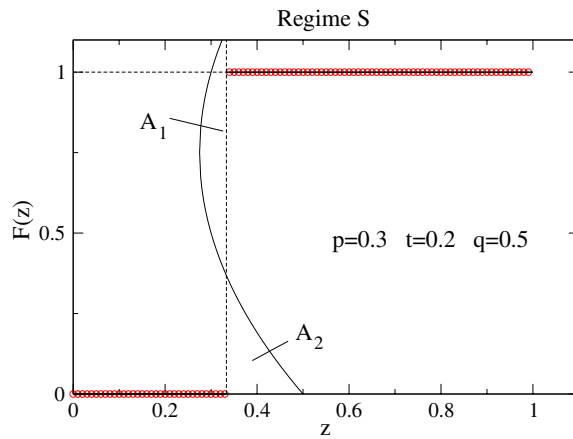
$$G'(\Phi) = z \quad (36)$$

as

$$\Phi_{\pm}(z) = \frac{q - \tau \pm [(q - \tau)^2 + (1 - 3\tau)(z - q)]^{1/2}}{1 - 3\tau}. \quad (37)$$



**Figure 5.** A representative graph for the solution in the regime  $C_s^+$ . The circles represent the data from the simulation of the microscopic model. The solid line is the analytical solution. The Maxwell construction which determines the location of the shock requires that the areas  $A_1$  and  $A_2$  be equal.



**Figure 6.** A representative graph for the solution in the regime  $S$ . The circles represent the data from the simulation of the microscopic model. The solid line is the analytical solution. The Maxwell reconstruction to determine the location of the shock requires that the areas  $A_1$  and  $A_2$  are equal.

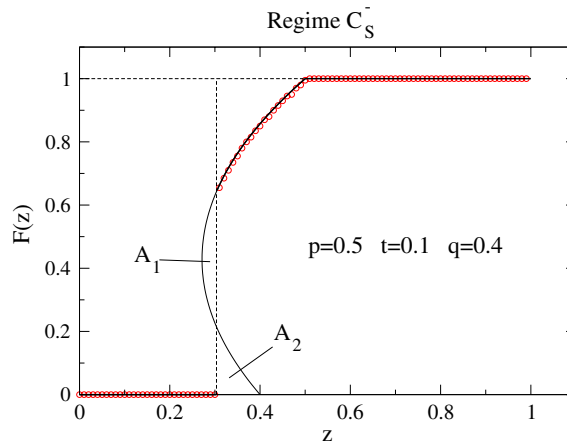
Note in particular that

$$\Phi_{\pm}(q) = \frac{q - \tau \pm |q - \tau|}{1 - 3\tau} \tag{38}$$

and

$$\Phi_{\pm}(p) = \frac{q - \tau \pm |p - \tau|}{1 - 3\tau}. \tag{39}$$

In regimes  $C^-$ ,  $C^0$  and  $C^+$  the function  $G'(F)$  increases monotonically from  $q$  to  $p$  as  $F$  goes from 0 to 1. This results in a continuous solution for  $q \leq z \leq p$ . In regions with the label



**Figure 7.** A representative graph for the solution in the regime  $C_S^-$ . The circles represent the data from the simulation of the microscopic model. The solid line is the analytical solution. The Maxwell reconstruction to determine the location of the shock requires that the areas  $A_1$  and  $A_2$  are equal.

$S$  on the other hand,  $G'(F)$  does not have a monotonous behavior and from the characteristic construction we see regions that are multiple valued. These cases are resolved by the Maxwell (or equal area) construction [22, 23], which is a consequence of the weak solution prescription based on equation (27), as we will describe below.

**Region  $C^-$  :**  $p > 1/3 > \tau \geq q$

Here we have  $\Phi_+(q) = 0$ ,  $\Phi_+(p) = 1$  and the solution  $F(x, \tau)$  is given by equation (35) with  $\Phi(z) = \Phi_+(z)$ ,  $z_l = q$  and  $z_r = p$ .

To understand the structure emerging from this solution, let us recall that the population density of agents  $f$  is given by the first derivative of  $F$ . The fact that  $F$  is a concave down and monotonically increasing function for this case implies that most of the agents are in the lower point range. Thus this situation is very different from the one encountered in the two-agent game, where the score distribution was uniform. One can argue that this is related to the fact that in this kind of three-agent game the ordering of winning probabilities constitutes a pure rich-get-richer competition and that once a team wins a game it actually beats two lower rank teams.

At this point it is useful to introduce two quantities in order to assess the score distributions emerging from the three-agent game. The ratio of the number of the poorest agents to the richest agents is related to the score distribution and is given by

$$\omega \equiv \frac{f(z_l)}{f(z_r)} = \frac{F'(z_l)}{F'(z_r)}. \tag{40}$$

Another useful quantity is the length of the score region over which the agents are distributed and which is associated with the total score shared by them,

$$\sigma \equiv z_r - z_l. \tag{41}$$

Since these metrics characterize the distributions of the score of agents within their society of players, we will refer to these metrics as ‘social’ indices. In the solution regime  $C^-$  we are

considering here they are given as

$$\omega_{C^-} = \sqrt{1 + \frac{(1 - 3t)(p - q)}{(q - t)^2}} > 1, \tag{42a}$$

$$\sigma_{C^-} = p - q. \tag{42b}$$

The fact that  $\omega_{C^-} > 1$  shows the bias in the distribution as opposed to the two-agent game. This bias is most extreme in the special case when  $q = t \Rightarrow p = 1 - 2t$ , so that  $\Phi$  becomes

$$\Phi(z) = \left[ \frac{z - t}{1 - 3t} \right]^{1/2} \tag{43}$$

and the ratio of poorest to richest agents  $\omega$  in the population diverges as a simple pole. This means that a great majority of the agents are near the poorest ones.

**Region  $C^0$  :**  $p > t = 1/3 > q$

We again have  $z_l = q, z_r = p$  and  $\Phi(z) = \Phi_+(z)$ , which simplifies to

$$\Phi(z) = \frac{z - q}{p - q} \tag{44}$$

for  $q \leq z \leq p$ .

Note that the resulting profile is qualitatively the same as the one for the two player game with  $q < p$ . This is so, because the middle team in a three player set has a 1/3 chance of winning the game and thus the team's score does not induce any bias toward either to the rich or poor side of the spectrum of agents. So 1/3 of the games can be considered to be completely redundant given  $p > t = 1/3 > q$ . For this case the social indices become

$$\omega_{C^0} = 1, \tag{45a}$$

$$\sigma_{C^0} = p - q, \tag{45b}$$

that is a uniform score distribution.

**Region  $C^+$  :**  $p \geq t > 1/3 > q$

We still have  $z_l = q, z_r = p$  and  $\Phi(z) = \Phi_+(z)$ . The social indices become

$$\omega_{C^-} = \sqrt{1 - \frac{|1 - 3t|(p - q)}{(q - t)^2}} < 1, \tag{46}$$

$$\sigma_{C^-} = p - q.$$

On the line  $t = p \Rightarrow q = 1 - 2t$ , we have in particular

$$\Phi(z) = 1 - \left[ 1 + \frac{z - q}{1 - 3q} \right]^{1/2}. \tag{47}$$

In this extreme case  $\omega_{C^-} = 0$ , meaning that the majority of the agents are condensed near the rich side  $z = p$ .

We therefore see the common property of the regions without a shock front: in all three regimes one has the ordering of probabilities  $p \geq t \geq q$  yielding a generally competitive game. The relative position of  $t$  with respect to 1/3 determines whether one has an accumulation of the agents toward the rich or poor side. When  $t = 1/3$  this ratio is equal. We present representative graphs of the regimes  $C^-, C^0$  and  $C^+$  in figures 2, 3 and 4, respectively. The figures compare

for a certain choice of parameters, both the score distribution obtained from simulation of the microscopical competitions as well as our analytical results in the hydrodynamical limit.

**Region  $C_3^+$  :  $t > p$  and  $q < 1/3$**

In this parameter regime  $G'(F)$  is not a monotonically increasing function in the interval  $[0, 1]$ . This leads to a formation of a shock front at  $z_r$ , where the latter is determined by the Maxwell equal area construction: the general solution in this region is again of the form (35) with  $z_l = q$ . The location of the shock  $z_r$  is determined from the equal area rule

$$G'[\Phi(z_r)] = \frac{G[1] - G[\Phi(z_r)]}{1 - \Phi(z_r)}. \tag{48}$$

After a little bit of algebra one finds that

$$\Phi(z_r) = \frac{1}{2} \frac{1 - 3q}{3t - 1}, \tag{49}$$

and using  $G'[\Phi(z_r)] = z_r$ , we obtain

$$z_r = q + \frac{1}{4} \frac{3q - 1}{1 - 3t} (4t - q - 1). \tag{50}$$

We see that at  $z_r$  we have a discontinuity that jumps from  $F = \Phi(z_r)$  to  $F = 1$ . The resulting  $\Phi$  profile is  $\Phi(z) = \Phi_+(z)$ , for  $z_l \leq z \leq z_r$  followed by a shock discontinuity at  $z_r$ .

For a single discontinuity it is clear that due to the equal area construction resulting in the vertical segment, the area under the graph  $G'(F)$  remains unchanged and one has

$$\begin{aligned} \frac{1}{3} &= G(1) - G(0) = \int_0^1 G'(F) dF \\ &= [1 - \Phi(z_r)]G'[\Phi(z_r)] + \int_0^{\Phi(z_r)} G'(F) dF, \end{aligned} \tag{51}$$

independent of  $p, q$  and  $t$ , since it is the rate at which the total number of points accumulates as each team participates in a three player game. In fact, the equal area construction resolves a region with multiple-valued points by a discontinuity such that the total conserved quantity remains unchanged. Thus for a single discontinuity this construction is equivalent to conserving the quantity over the whole domain as utilized in [21].

On the other hand, when one has four or higher agent games, depending on the winning probabilities, the resulting profiles can have multiple shocks separated by rarefaction waves. In that case a global conservation constraint will not be sufficient to determine all shock locations, and one will have to resort to the use of the equal area construction to each region where the profile is multiple valued.

One can qualitatively understand the reason for a shock front to the right of the wealth span as follows. First of all, due to the fact that  $t > p$ , having the highest score in a competition is a disadvantage since a team in the middle of a triplet is favored. This results in a deceleration mechanism for the propagation rate of teams with high scores and consequently a fraction of the total agents condense at the high score side of the spectrum: they are the richest agents in the society. It is clear that such a case is an attractor solution since when it is formed it is not destroyed. That is, if an agent in this shock wins a game it will be disfavored in the future games resulting in a loss of its point. Conversely, if an agent in the shock loses a game it will be favored in a future game over a team in the shock resulting in its return to the shock region.

The region with a continuous  $F$  to the left of the shock can be understood by noting that  $q < 1/3$ , which implies  $q < (t + p)/2$ , regardless of how  $p$  and  $q$  are ordered. Let us consider a game with two teams from the shock region and one from below, such a game constitutes a great majority of possible types of games if there is a shock region. In this case the probability of a win for the teams in the shock is given by  $(t + p)/2$  and for the other by  $q$ . This means that the lower point team is disfavored altogether and the criterion for this is just  $q < 1/3$ , resulting in a continuous population density for regions below the shock.

Due to the discontinuity in  $F$  one has to exercise a little more care in implementing the social index  $\omega$  we have introduced before, since in this case we have

$$F(z) = \Phi_+(z) + [1 - \Phi_+(z_r)] \Theta(z - z_r),$$

where  $\Theta(z)$  is the step function. Thus there will be a strong singularity in  $\omega$

$$\omega_{C_s^+} = \frac{\text{const}}{\delta(0)}, \tag{52a}$$

$$\sigma_{C_s^+} = \frac{4t - q - 1}{4} \left( \frac{1 - 3q}{3t - 1} \right). \tag{52b}$$

So  $\omega$  vanishes much more strongly (and for all parameters) than in the particular case presented for the regime  $C^+$  where the vanishing was determined by a simple zero.

A representative graph including an exact numerical simulation of the microscopic model and the analytical construction is presented in figure 5.

**Region  $S$  :  $q \geq 1/3 > p$**

In this region  $G'[F]$  is monotonically decreasing with  $F$ . The profile  $\Phi$  is obtained from the equal area construction, resulting in a discontinuity at  $z^*$  that covers the whole interval  $[0, 1]$  of  $F$ , i.e. a unit step. The shock speed is found to be  $1/3$  so that  $z_l = z_r = z^* = 1/3$ . Note that when  $p > t$  and  $p \leq 1/3$ ,  $G'[F]$  is no longer monotonically decreasing in  $F$ . However the equal area construction still leads to a discontinuity at  $z$  covering the whole interval of  $F$  so that the shock speed is again  $1/3$  and thus  $z^* = 1/3$ .

The social indices become  $\omega_S = 1$  and  $\sigma_S = 0$ , re-emphasizing the fact that in this case all agents share the same wealth.

The meaning of a single shock becomes clear if we note that  $q > p$  implies that the lower score teams will eventually catch up with the teams with highest scores. The other condition  $q > 1/3$  implies  $q > (t + p)/2$  which means that the lower rank teams will also be able to catch up with a shock to the right of the wealth span. Note that  $q > p$  does not necessarily imply  $q > t$ , although there is a subregion consistent with this condition. The condition  $q > (t + p)/2$  ensures that even if  $t > p$  and  $t > q$ , whenever there is a shock to the right as argued in our discussion of the region  $C_s^+$ , the lower rank teams will eventually reach that shock resulting in a single discontinuous front for the solution.

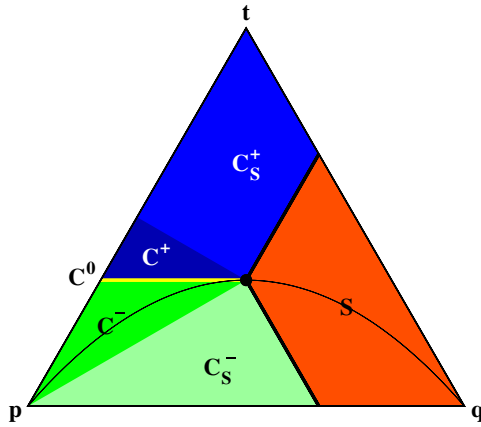
A representative graph including an exact numerical simulation of the microscopic model and the analytical construction is presented in figure 6.

**Region  $C_s^-$  :  $q > t$  and  $p > 1/3$**

In this parameter regime  $G'(F)$  is not a monotonically increasing function in the interval  $[0, 1]$ . This leads to a formation of a shock front at  $z_l$ , where the latter is determined by the Maxwell equal area construction: the general solution in this region is again of the form (35) with  $z_r = p$ . The location  $z_l$  of the shock is determined by the equal area rule as

$$G'[\Phi(z_l)] = \frac{G[\Phi(z_l)] - G[0]}{\Phi(z_l)} \tag{53}$$





**Figure 8.** (Colour online) The phase diagram of the three-agent game presented on the plane  $p + t + q = 1$ . The blue/green areas represent the regimes with positive/negative concavity of  $F$  (regions whose labels have a +/-). The red area, labelled S, is where the solution is a single shock wave at  $z = \tau/3$ . The regions  $C_S^+$  and  $C_S^-$  are represented by the lighter shades of blue/green. The crossover line from  $C^-$  to  $C^+$  is emphasized with a yellow line. The thin black curve represents the resolution of a three-agent game via a mini-tournament between the three players in terms of two-agent competitions as described in the text.

with

$$\Phi(z_l) = \frac{3}{2} \frac{q - t}{1 - 3t} \tag{54}$$

so that

$$z_l = q - \frac{3}{4} \frac{(q - t)^2}{1 - 3t}. \tag{55}$$

The resulting profile is given by  $\Phi(z) = \Phi_+(z)$ , for  $z_l \leq z \leq z_r$  preceded by a shock discontinuity at  $z_l$  that jumps from  $F = 0$  to  $F = \Phi(z_l)$ .

The qualitative analysis of this solution is very similar to our previous discussion of regimes with a shock. The condition  $q > t$  means that the lower rank teams will catch up with the middle rank teams. The condition  $p > 1/3$  gives  $p > (t + p)/2$  meaning that higher point teams can decouple from this region and constitute a continuous solution to the right of the shock.

The social indices become

$$\omega_{C_S^-} = \text{const} \times \delta(0) \tag{56}$$

$$\sigma_{C_S^-} = p - q + \frac{3}{4} \frac{(q - t)^2}{1 - 3t}. \tag{57}$$

The index  $\omega_{C^-}$  diverges much more strongly (and for all parameters) as opposed to the simple pole divergence we have seen for the particular case of parameters in regime  $\omega_{C^-}$ .

A representative graph including an exact numerical simulation of the microscopic model and the analytical construction is presented in figure 7.

In figure 8 we have graphically combined all the regimes in a phase diagram on the  $p + q + t = 1$  plane. The vertices  $p, q$  and  $t$  of the triangle correspond to the cases  $p = 1, q = 1$  and  $t = 1$ , respectively, while the edges  $\overline{pt}, \overline{tq}$  and  $\overline{qp}$ , correspond to  $q = 0, p = 0$

and  $t = 0$ , respectively. The points  $(p, q, t)$  with  $p$  constant correspond to lines parallel to the edge  $\overline{tq}$ , etc, and the dot shown represents the point  $(1/3, 1/3, 1/3)$ .

### 3.3. Social structures emerging from different regimes

If we regard the agent game as a social model where agents compete for an (unlimited) commodity through three-way competitions, we see that, depending on  $(p, q, t)$ , different wealth structures emerge asymptotically. Following an analog of the naming scheme presented in [20], we can classify the resulting agent societies in terms of their wealth distributions as follows.

- Regime  $C^-$ : middle-class society with mild hierarchy.
- Regime  $C^0$ : pure middle-class society.
- Regime  $C^+$ : middle-class society with mild anti-hierarchy.
- Regime  $C_S^+$ : anti-hierarchical society.
- Regime  $S$ : egalitarian society.
- Regime  $C_S^-$ : hierarchical society.

In all regions without a shock, the agents are distributed along the same wealth span  $(p - q)t$ . The reason we endow all of them with a middle-class structure is due to the fact that  $F$  is a continuous function and thus has no Dirac-delta singularities in its derivative, i.e. there are no divergent condensations of agents at the highest or lowest points.

The regimes with a shock on the other hand constitute genuinely different societies where some agents condense at a given point accumulation rate. This implies analogies to social structures where there is a frozen class. The terms anti-hierarchy/hierarchy mean in our context that the condensation happens in the upper/lower part of the society. The term egalitarian denotes the case where all agents increase their points at the same rate.

The regimes with a shock have another common feature: unlike the regimes without a shock the parameters *do not* obey  $p \geq t \geq q$ ; that is the game is not purely competitive. For instance, if the lower rank agents are favored over the higher rank ones, such as in regime  $S$ , we might make an analogy with a sort of welfare system which in the long run results in an equal wealth distribution. In a regime where the middle rank agents are favored over the higher rank ones, regime  $C_S^+$  is suggestive of some form of affirmative action. In regime  $C_S^-$  the lower rank is favored over the middle rank and the ultimate outcome is to coalesce these regimes resulting in a large volume of poor agents. This corresponds to a type of affirmative action where the middle class is the sole source of wealth flux to the lower class. An upper class also persists in this regime and the resulting structure can therefore be called highly hierarchical.

As evident from our analysis of the different regimes, a particular  $p, t, q$  ordering does not necessarily ensure that the solution will be in a unique regime. For regimes with a shock, we show below whether a particular ordering resides in a single regime or not:

- (1)  $p > q > t$ : single regime:  $C_S^-$ .
- (2)  $q > p > t$ : two regimes:  $C_S^-$  or  $S$ .
- (3)  $q > t > p$ : single regime:  $S$ .
- (4)  $t > q > p$ : two regimes:  $S$  or  $C_S^+$ .
- (5)  $t > p > q$ : single regime:  $C_S^+$ .

## 4. Extensions of the three-agent game

In this section we consider two variations of the three-agent game introduced. We first consider each three-agent game as a mini-tournament of two player games among the participating three

agents, with the tournament winner winning the three-agent game. We then turn to the case considered in [20], where a uniform score decay rate is introduced and show how this case is readily solved in the hydrodynamic limit using the method of characteristics.

#### 4.1. Resolution in terms of a two-agent game

A three-agent game will be characterized by the set of numbers  $(p, t, q)$  which controls the outcome of a single competition. However it is also possible to resolve a three-agent games in terms of two-agent games. The simplest such approach is to let all three teams play a single two-agent game with each other, that is to have a tournament. All the two-agent games in this tournament are decided based on how many tournaments the agents have won before. That is, during the tournament accumulated scores of each team are frozen, but they accumulate match points depending on the tournament winner. The winner of the tournament is the agent with largest number of accumulated match points in that tournament and he scores a tournament point. As usual ties are decided on the basis of equal likelihood. A two-agent game is characterized by two numbers  $P + Q = 1$ , with  $P$  denoting the probability that the team with the larger number of points, out of the two, wins. A simple analysis yields the following probabilities to emerge as the winner out of such a tournament

$$p = P^2 + \frac{1}{3}PQ \quad (58a)$$

$$t = \frac{4}{3}PQ \quad (58b)$$

$$q = Q^2 + \frac{1}{3}PQ. \quad (58c)$$

These are represented in the phase diagram for the three-agent game in figure 8. As one might expect, for  $P > 1/2$  one is in the  $C^-$  regime. The value  $P = 1/2$  means that  $p = t = q = 1/3$  and we have a completely random game with a score distribution represented by a single shock. For values of  $P$  less than  $1/2$  the shock remains as in the two-agent game and in the terminology of the three-agent phase diagram we are in regime  $S$ . Note that the  $C_S^-, C^0, C^+$  and  $C_S^+$  regimes do not occur in this variation of the three-agent game.

#### 4.2. Adding a decline rate for agents

An interesting extension of the model we presented is to allow for a mechanism with which agents lose points. This for instance can account for the realistic observation that through inaction competitors lose fitness. A simple realization of this is to allow a *steady* and democratic decline rate for agents, as advocated and studied in detail for two-agent games in [20]. Such a decline rate for agents will simply result in the following generalization of (11):

$$\frac{\partial F}{\partial \tilde{\tau}} = -\frac{\partial F}{\partial \tilde{x}}[r + G'(F)], \quad (59)$$

where  $r$  is the rate with which agents lose points.

This new form of the equation is entirely related to the old one by a Galilean transformation

$$\tilde{\tau} = \tau \quad (60a)$$

$$\tilde{x} = x + r\tau. \quad (60b)$$

One can therefore use the solutions of the old equation (11) to generate solutions for the new one by just left translating the  $x$  axis. This is analogous to just left shifting the  $z$  variable by  $r$ . However even though the equation is Galilean invariant the boundary conditions are not:

we do not allow for negative points for agents. Therefore the recipe for generating solutions for the advance-decline model is to take a solution without  $r$ , left translate the  $z$  axis by  $r$  and discard the solution for  $z < 0$ . This will possibly generate condensation of agents at zero points, and hence a shock at  $z = 0$ .

Presented this way, adding a decline rate is straightforward and the asymptotic regimes we have presented will in general double in number. Whether a shock at  $z = 0$  exists or not can easily be determined by comparing  $r$  and  $z_l$ . If  $r > z_l$  there will be a shock at  $z = 0$ , if not, the solution will qualitatively look the same.

As far as the decline rate is concerned, the most interesting regime is  $C_S^+$  where there is originally a shock at  $z_r$ , so in this case if  $r < q$  we will have two shocks. For  $C_S^-$  the effect of  $r$  is qualitatively immaterial since the shock is already to the left of the curve. This is also valid for regime  $S$ .

In terms of the social classification we have presented, the effect of  $r$  is to possibly introduce additional hierarchy into the society, by allowing for the condensation of agents at the  $x = 0$  end of the score range, constituting those 'left behind'.

## 5. Discussion

We have studied the complete dynamics of the three-agent game in the hydrodynamical limit of large scores and large number of games played, by noting that the cumulative score distribution obeys a hyperbolic conservation law PDE, and using the method of characteristics to obtain analytical solutions.

The applications of this model to realistic social data would be rather interesting. The effect of a policy reminiscent of some sort of affirmative action can also be applied in our model in that it admits three players.

One possible extension of our model that has potentially interesting implications within the competitive subspace is the merger option of two lower ranking agents in a game. That is two agents combining forces against the more powerful opponent. In doing so they should both increase their probabilities of winning in comparison to the case without merger.

Another interesting avenue is to increase the number of attributes in choosing a winner. This will in general mean that the rate equations we have used will involve a multi-dimensional gradient representing the different attributes. These extensions are currently under study.

## Acknowledgments

We thank K Fortuny for the careful reading of this manuscript. The simulations needed for this work have been performed on Gilgamesh, the computer cluster of the Feza Gürsey Institute.

## References

- [1] Weidlich W 2000 *Sociodynamics: A Systematic Approach to Mathematical Modelling in the Social Sciences* (New York: Harwood Academic)
- [2] Helbing D, Farkas I and Vicsek T 2000 *Nature* **407** 487
- [3] Couzin I D, Krause J, Franks N R and Levin S A 2005 *Nature* **433** 513
- [4] Ispolatov S, Krapivsky P L and Redner S 1998 *Eur. Phys. J. B* **2** 267
- [5] Dragulescu A and Yakovenko V M 2000 *Eur. Phys. J. B* **17** 723
- [6] Slanina F 2004 *Phys. Rev. E* **69** 046102
- [7] Ree S 2006 *Phys. Rev. E* **73** 026115
- [8] Ben-Naim E and Hengartner N W 2006 *Preprint physics/0612217*
- [9] Ben-Naim E, Vazquez F and Redner S 2006 *J. Quant. Anal. Sports* **2** Article 1

- [10] Weisbuch G, Deffuant G, Amblard F and Nadal J P 2002 *Complexity* **7** 55
- [11] Ben-Naim E, Krapivsky P L and Redner S 2003 *Physica D* **183** 190
- [12] Stauffer D and Meyer-Ortmanns H 2004 *Int. J. Mod. Phys. B* **15** 241
- [13] Wasserman S and Faust K 1994 *Social Network Analysis* (Cambridge: Cambridge University Press)
- [14] Girvan M and Newman M E J 2002 *Proc. Natl Acad. Sci. USA* **99** 7821
- [15] Zannette D 2002 *Phys. Rev. E* **65** 041908
- [16] Bonabeau E, Theraulaz G and Deneubourg J-L 1995 *Physica A* **217** 373
- [17] Sousa A O and Stauffer D 2000 *Int. J. Mod. Phys. C* **11** 1063
- [18] Malarz K, Stauffer D and Kulakowski K 2005 *Preprint physics/0502118*
- [19] Ben-Naim E and Redner S 2005 *J. Stat. Mech.* **L11002**
- [20] Ben-Naim E, Vazquez F and Redner S 2006 *Eur. Phys. J. B* **49** 531
- [21] Ben-Naim E, Kahng B and Kim J S 2006 *J. Stat. Mech.* **P07001**
- [22] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)
- [23] LeVeque R J 1992 *Numerical Methods and Conservation Laws* 2nd edn (Basel: Birkhäuser)